The chiral anomaly-from path integral method to differential geometric approach

1988 J. Phys. A: Math. Gen. 21 1811

(http://iopscience.iop.org/0305-4470/21/8/017)

J. Phys. A: Math. Gen. 21 (1988) 1811-1829. Printed in the UK

The chiral anomaly—from path integral method to differential geometric approach

Rong-tai Wang and Guang-jiong Ni
Modern Physics Institute, Fudan University, Shanghai, China

Received 30 June 1987, in final form 23 November 1987

Abstract. As a continuation of a previous investigation, the exterior differential formulae for non-Abelian anomalies are derived within the reformulated path integral scheme, and a number of differential geometric objects are traced out reversely. With the help of these objects, a compact form for the gauged Wess-Zumino effective action induced by finite chiral transformation is also derived in the path integral formalism.

1. Introduction

In a previous paper (Wang and Ni 1987, hereafter referred to as I), we presented a new formulation of a path integral scheme for applying Fujikawa's idea on deriving chiral anomaly (Fujikawa 1979, 1980, 1986) to more general Abelian and non-Abelian cases with γ_5 coupling in $2n$ dimensions. Here, based on this reformulated path integral scheme, we are going to exhibit the detailed derivation of the more compact formulae for chiral anomaly in exterior differential forms which usually appear in the literature of the geometric approach (see, for example, Zumino *et al* 1984, Kawai *et al* 1984, Alvarez-Gaumé and Ginsparg 1984, 1985, Jackiw 1985). Within this path integral formalism we also make some investigations into the Wess-Zumino effective action which was originally derived by Witten (1983) on the basis of topological considerations and later constructed by some authors in a purely differential geometric way (Kawai *et al* 1984, Chou *et al* 1984, Ingermanson 1985, Mañes 1985). As is well known, the differential geometric approach concerning the mathematical structure of non-Abelian anomalies is a powerful non-perturbative method. However, we can still pose ourselves the following questions. (a) Can we derive these exterior differential expressions for anomalies using a path integral method? (b) How can we investigate the Wess-Zumino action within a path integral scheme? In short, is it possible to find a theory which may provide us with a connection between path integral derivation and differential geometric technique? We will try to answer these questions in this paper. We will also show that all the objects which appeared in the usual procedure from Chern character to Chern-Simons for deriving anomalies in a differential geometric approach can now be traced out reversely in our formalism of path integral. The organisation of this paper is as follows. In § 2, some key points and main results of this reformulated path integral scheme are reviewed. In § 3, starting with a general formula obtained in I, the derivations of the exterior differential formulae for the chiral anomaly in the case of chiral gauge coupling and the gauge anomaly are exhibited in detail. In § 4, the differential geometric objects such as topological invariant ω_{2n+1} forms for anomalies and Bardeen counter-terms are traced out systematically. In § 5 the treatments of finite chiral transformation in this reformulated path integral scheme are explained in detail and thereby the Wess-Zumino effective action is derived in a compact manner.

2. The reformulated path integral scheme for deriving chiral anomalies

The phenomenon where a symmetry with its related conserved current inferred by Noether's theorem at the classical level is spoiled by the procedure of quantisation is generally referred to as anomaly. The basic observation of the path integral approach to the chiral anomaly is that the path integral measure is non-invariant under chiral transformation (Fujikawa 1979, 1980a, b, 1986). In this section, we give a short illustration about the reformulated path integral scheme presented in I by considering the general fermion field theory with γ_5 coupling defined in $2n$-dimensional Euclidean space

$$W(V, A) = \int \mathrm{d}\mu(\psi) \exp\left(-\int \mathrm{d}x\, \bar{\psi} \mathrm{i} \not{D} \psi\right) \tag{2.1}$$

where

$$\mathrm{i}\not{D} = \mathrm{i}(\not{\partial} + \not{V} + \not{A}\gamma_5) \tag{2.2}$$

$$V = V^a_\mu \lambda^a \qquad A = A^a_\mu \lambda^a \tag{2.3}$$

with V^a_μ, A^a_μ being external gauge fields, λ^a the anti-Hermitian generator of G and $\mathrm{d}\mu(\psi)$ the path integral measure

$$\mathrm{d}\mu(\psi) = \prod_x \mathscr{D}\bar{\psi}(x) \mathscr{D}\psi(x). \tag{2.4}$$

2.1. *Anomalous factor*

As noted by Fujikawa, the measure (2.4) is non-invariant under the infinitesimal chiral transformation specified by $\beta(x) = \beta^a(x)\lambda^a$

$$\begin{aligned} \psi(x) &\to \psi'(x) = \exp(\beta(x)\gamma_5)\psi(x) \\ \bar{\psi}(x) &\to \bar{\psi}'(x) = \bar{\psi}(x)\exp(\beta(x)\gamma_5). \end{aligned} \tag{2.5}$$

This implies

$$\begin{aligned} \int \mathrm{d}\mu(\psi) \exp\left(-\int \mathrm{d}x\, \bar{\psi}\mathrm{i}\not{D}\psi\right) &\equiv \int \mathrm{d}\mu(\psi') \exp\left(-\int \mathrm{d}x\, \bar{\psi}'\mathrm{i}\not{D}\psi'\right) \\ &\neq \int \mathrm{d}\mu(\psi) \exp\left(-\int \mathrm{d}x\, \bar{\psi}'\mathrm{i}\not{D}\psi'\right). \end{aligned} \tag{2.6}$$

Now, by denoting

$$\mathrm{i}\not{D}(\beta) = \exp(\beta\gamma_5)\mathrm{i}\not{D}\exp(\beta\gamma_5) \tag{2.7}$$

$$W(V, A; \beta) = \int \mathrm{d}\mu(\psi) \exp\left(-\int \mathrm{d}x\, \bar{\psi}\mathrm{i}\not{D}(\beta)\psi\right) \tag{2.8}$$

and $\exp\,\delta\Gamma(\beta)$ as the anomalous factor with

$$\delta\Gamma(\beta) = -\int \mathrm{d}x\, \beta(x) G(x) \tag{2.9}$$

we would have an equality

$$W(V, A) = \exp(\delta\Gamma(\beta))\, W(V, A, \beta). \tag{2.10}$$

Expanding (2.10) in powers of $\beta^a(x)$, one obtains the anomalous axial current identity

$$D_\mu \langle \bar{\psi}\gamma^\mu\gamma_5\lambda^a\psi\rangle = G^a(x). \tag{2.11}$$

Equation (2.10) leads to an expression for the anomalous factor as the ratio of two fermion determinants

$$\mathrm{e}^{-\delta\Gamma(\beta)} = \frac{W(V, A;\, \beta)}{W(V, A)} = \frac{\det \mathrm{i}\not{D}(\beta)}{\det \mathrm{i}\not{D}}. \tag{2.12}$$

Furthermore, after s steps of infinitesimal chiral transformation, the equality (2.10) becomes

$$W(V, A) = \exp\left(\sum_{r=1}^{s} \delta\Gamma(V, A;\, \beta_1, \ldots, \beta_r)\right) W(V, A;\, \beta_1, \ldots, \beta_s) \tag{2.13}$$

where

$$\begin{aligned}\exp[-\delta\Gamma(\beta_1, \ldots, \beta_r)] &= \frac{W(V, A;\, \beta_1, \ldots, \beta_r)}{W(V, A;\, \beta_1, \ldots, \beta_{r-1})}\\ &= \frac{\det \mathrm{i}\not{D}(\beta_1, \ldots, \beta_r)}{\det \mathrm{i}\not{D}(\beta_1, \ldots, \beta_{r-1})}.\end{aligned} \tag{2.14}$$

2.2. *The comoving representation*

Since the Dirac operator $\mathrm{i}\not{D}(\beta_1, \ldots, \beta_r)$ is non-Hermitian in general, two Hermitian operators are introduced:

$$\Delta(\beta_1, \ldots, \beta_r) \equiv [\mathrm{i}\not{D}(\beta_1, \ldots, \beta_r)]^\dagger [\mathrm{i}\not{D}(\beta_1, \ldots, \beta_r)] \tag{2.15}$$

$$\tilde{\Delta}(\beta_1, \ldots, \beta_r) \equiv [\mathrm{i}\not{D}(\beta_1, \ldots, \beta_r)][\mathrm{i}\not{D}(\beta_1, \ldots, \beta_r)]^\dagger \tag{2.16}$$

with eigenequations

$$\begin{aligned}\Delta(\beta_1, \ldots, \beta_r)\phi_n(x;\, \beta_1, \ldots, \beta_r) &= \lambda_n^2 \phi_n(x;\, \beta_1, \ldots, \beta_r)\\ \tilde{\Delta}(\beta_1, \ldots, \beta_r)\tilde{\phi}_n(x;\, \beta_1, \ldots, \beta_r) &= \lambda_n^2 \tilde{\phi}_n(x;\, \beta_1, \ldots, \beta_r)\end{aligned} \tag{2.17}$$

and mapping relations for $\lambda_n \neq 0$

$$\begin{aligned}{}[\mathrm{i}\not{D}(\beta_1, \ldots, \beta_r)]\phi_n(x;\, \beta_1, \ldots, \beta_r) &= \lambda_n \exp[\mathrm{i}\theta_n(\beta_1, \ldots, \beta_r)]\tilde{\phi}_n(x;\, \beta_1, \ldots, \beta_r)\\ [\mathrm{i}\not{D}(\beta_1, \ldots, \beta_r)]^\dagger\tilde{\phi}_n(x,;\, \beta_1, \ldots, \beta_r) &= \lambda_n \exp[-\mathrm{i}\theta_n(\beta_1, \ldots, \beta_r)]\phi_n(x;\, \beta_1, \ldots, \beta_r).\end{aligned} \tag{2.18}$$

So, the Dirac operator $\mathrm{i}\not{D}(\beta_1, \ldots, \beta_r)$ is diagonalised in the representation with $\{\phi_n(x;\, \beta_1, \ldots, \beta_r)\}$ as its right bases and $\{\tilde{\phi}_n(x;\, \beta_1, \ldots, \beta_r)\}$ as its left bases

$$\int \mathrm{d}x\, \tilde{\phi}_m^\dagger(x;\, \beta_1, \ldots, \beta_r)\mathrm{i}\not{D}(\beta_1, \ldots, \beta_r)\phi_n(x;\, \beta_1, \ldots, \beta_r) = \lambda_n\, \mathrm{e}^{\mathrm{i}\theta_n(\beta_1, \ldots, \beta_r)}\delta_{mn}. \tag{2.19}$$

Now denoting

$$\begin{aligned}\delta\mathrm{i}\not{D}(\beta_1, \ldots, \beta_r) &= \mathrm{i}\not{D}(\beta_1, \ldots, \beta_r) - \mathrm{i}\not{D}(\beta_1, \ldots, \beta_{r-1})\\ &= \{\mathrm{i}\not{D}(\beta_1, \ldots, \beta_{r-1}), \beta_r\gamma_5\}\end{aligned} \tag{2.20}$$

and treating it as a perturbation, one easily obtains its first-order correction to the diagonal element with the help of mapping relations (2.18), i.e.

$$\int \mathrm{d}x\, \tilde{\phi}_n^{\dagger}(x;\beta_1,\ldots,\beta_{r-1})\delta \mathrm{i}\not{D}(\beta_1,\ldots,\beta_r)\phi_n(x;\beta_1,\ldots,\beta_{r-1})$$

$$= \lambda_n \exp[\mathrm{i}\theta_n(\beta_1,\ldots,\beta_{r-1})] \int \mathrm{d}x\, \tilde{\phi}_n^{\dagger}(x;\beta_1,\ldots,\beta_{r-1})$$

$$\times \beta_r\gamma_5\tilde{\phi}_n(x;\beta_1,\ldots,\beta_{r-1}) + \lambda_n \exp[\mathrm{i}\theta_n(\beta_1,\ldots,\beta_{r-1})]$$

$$\times \int \mathrm{d}x\, \phi_n^{\dagger}(x;\beta_1,\ldots,\beta_{r-1})\beta_r\gamma_5\phi_n(x;\beta_1,\ldots,\beta_{r-1}). \tag{2.21}$$

This implies the diagonal element of the Dirac operator would acquire an additional phase factor $\exp[\mathrm{i}\delta\theta_n(\beta_1,\ldots,\beta_r)]$ with

$$\mathrm{i}\delta\theta_n(\beta_1,\ldots,\beta_r) = \int \mathrm{d}x\, \tilde{\phi}_n^{\dagger}(x;\beta_1,\ldots,\beta_{r-1})\beta_r\gamma_5\tilde{\phi}_n(x;\beta_1,\ldots,\beta_{r-1})$$

$$+ \int \mathrm{d}x\, \phi_n^{\dagger}(x;\beta_1,\ldots,\beta_{r-1})\beta_r\gamma_5\phi_n(x;\beta_1,\ldots,\beta_{r-1}) \tag{2.22}$$

and one can express

$$\delta\Gamma(\beta_1,\ldots,\beta_r) = \sum_n \mathrm{i}\delta\theta_n(\beta_1,\ldots,\beta_r) \tag{2.23}$$

with $\mathrm{i}\delta\theta_n(\beta_1,\ldots,\beta_r)$ shown as (2.22). Note that, as pointed out in I, expression (2.22) is correct even for the zero diagonal elements. What we wish to stress here is that, although the similar mapping relations (2.18) were familiar in previous literature, the phase factors $\exp(\mathrm{i}\theta_n)$ were often overlooked. In fact, it is the accumulating effect of this additional argument change which leads to the appearance of the anomaly.

2.3. *The null space regularisation scheme and a general formula for chiral anomaly*

We also explored in I the topological implication of Wess–Zumino consistent conditions (Wess and Zumino 1971) which impose on these additional phase factors. Let Δ_β be the coboundary operator. One then finds that the argument change of non-zero diagonal element $\delta\theta_n(\beta)$ obeys $\Delta_\beta\, \delta\theta_n(\beta) = 0$ trivially, whereas for zero diagonal element $\Delta_\beta\, \delta\theta_n(\beta) = \pm 2\pi$, the sign '$\pm$' depends on the chirality of the zero mode of $\mathrm{i}\not{D}$ or $(\mathrm{i}\not{D})^{\dagger}$. Considering this 1-cocycle property, we proposed a new regularisation scheme with only the argument change of zero diagonal elements preserved. This null space regularisation scheme was realised by the following expressions:

$$-\delta\Gamma(\beta) = \{\mathrm{Tr}\, \beta\gamma_5 f(\xi, \mathrm{i}\not{D})\}_{\xi\ \mathrm{independent}} + \{\mathrm{Tr}\, \beta\gamma_5 f(\xi, (\mathrm{i}\not{D})^{\dagger}\}_{\xi\ \mathrm{independent}} \tag{2.24}$$

where

$$f(\xi, \mathrm{i}\not{D}) \equiv \frac{1}{2\pi\mathrm{i}} \oint_c \frac{\mathrm{e}^{-\xi z}}{z - \mathrm{i}\not{D}}\, \mathrm{d}z \tag{2.25}$$

with c a path on complex z plane which bypasses the origin as shown in a figure in I. It should be pointed out that our regularisation scheme shown in (2.24) does satisfy the Wess–Zumino consistency conditions. After a careful calculation we arrive at a general formula for the regularised $\delta\Gamma(\beta)$

$$\delta\Gamma(V, A; \beta) = \frac{1}{(4\pi)^n n!} \sum_{m=0}^{n} B(m+1, n+1) \times \operatorname{Tr}\left\{\beta\gamma_5 \frac{1}{(2m)!}\left[\frac{d^{2m}}{dz^{2m}} \sum_{l_1+l_2=m+n} (Q_+^{l_1} Q_-^{l_2} + Q_-^{l_1} Q_+^{l_2})\right]_{z=0}\right\} \tag{2.26}$$

where

$$Q_\pm = F_{\mu\nu}^{(V)}\sigma_{\mu\nu} \pm F_{\mu\nu}^{(A)}\sigma_{\mu\nu} + 2z\not{A}\gamma_5 \tag{2.27}$$

with

$$\begin{aligned} F_{\mu\nu}^{(V)} &= \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu] + [A_\mu, A_\nu] \\ F_{\mu\nu}^{(A)} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, V_\nu] + [V_\mu, A_\nu] \end{aligned} \tag{2.28}$$

and

$$\sigma_{\mu\nu} = \tfrac{1}{4}[\gamma_\mu, \gamma_\nu] \qquad B(m+1, n+1) = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}. \tag{2.29}$$

As an example, the Bardeen anomaly in four-dimensional Euclidean space is (Bardeen 1969)

$$\begin{aligned} \delta\Gamma(V, A; \beta) = \frac{1}{4\pi^2} \operatorname{Tr}\{\beta(x)\varepsilon_{\mu\nu\rho\sigma}[F_{\mu\nu}^{(V)}F_{\rho\sigma}^{(V)} + \tfrac{1}{3}F_{\mu\nu}^{(A)}F_{\rho\sigma}^{(A)} + \tfrac{32}{3}A_\mu A_\nu A_\rho A_\sigma \\ - \tfrac{8}{3}(F_{\mu\nu}^{(V)}A_\rho A_\sigma + A_\mu F_{\nu\rho}^{(V)}A_\sigma + A_\mu A_\nu F_{\rho\sigma}^{(V)}]\}. \end{aligned} \tag{2.30}$$

3. The exterior differential formulae for non-Abelian anomaly

As a continuation of previous investigations we are going to derive the more compact formulae for non-Abelian anomalies based on the general result (2.26).

3.1. Bardeen anomaly in the case of chiral gauge coupling

The Dirac operator in the case of chiral gauge coupling is

$$i\not{D} = i\not{\partial} + i\not{A}^{(+)}\tfrac{1}{2}(1+\gamma_5). \tag{3.1}$$

Being a special case with

$$V_\mu = A_\mu = \tfrac{1}{2}A_\mu^{(+)} \tag{3.2}$$

the $Q_\pm$ in (2.27) becomes

$$Q_\pm = F_{\mu\nu}^{(+)}\sigma_{\mu\nu}\hat{p}_\pm + z\not{A}^{(+)}\gamma_5 \tag{3.3}$$

where

$$F_{\mu\nu}^{(+)} = \partial_\mu A_\nu^{(+)} - \partial_\nu A_\mu^{(+)} + [A_\mu^{(+)}, A_\nu^{(+)}] \tag{3.4}$$

$$\hat{p}_\pm = \tfrac{1}{2}(1 \pm \gamma_5). \tag{3.5}$$

We observe that only the terms $(Q_+^{l_1}Q_-^{l_2}+Q_-^{l_1}Q_+^{l_2})$ in the summation of the formula (2.26) with $n\leqslant l_1+l_2\leqslant 2n$ may have contributions to $\delta\Gamma$. Other terms such as $l_1+l_2>2n$ or $0\leqslant l_1+l_2<n$ will drop out under the combined operations of

$$\operatorname{Tr}\beta\gamma_5(*) \qquad \text{and} \qquad \left.\frac{\mathrm{d}^{2m}(*)}{\mathrm{d}z^{2m}}\right|_{z=0}$$

so one has the following equivalent expression (Gipson 1986):

$$\operatorname{Tr}\left\{\beta\gamma_5\left[\frac{\mathrm{d}^{2m}}{\mathrm{d}z^{2m}}\sum_{l_1+l_2=m+n}(Q_+^{l_1}Q_-^{l_2}+Q_-^{l_1}Q_+^{l_2})\right]_{z=0}\right\}$$
$$=\operatorname{Tr}\left\{\beta\gamma_5\left[\frac{\mathrm{d}^{2m}}{\mathrm{d}z^{2m}}\left(\frac{1}{1-Q_+}\frac{1}{1-Q_-}+\frac{1}{1-Q_-}\frac{1}{1-Q_+}\right)\right]_{z=0}\right\}. \tag{3.6}$$

Using the formulae of chiral expansion

$$\begin{aligned}(1-F_{\mu\nu}\sigma_{\mu\nu}\hat{p}_++A\!\!\!/\gamma_5)^{-1}&=\hat{p}_-+(1-A\!\!\!/)(1-F_{\mu\nu}\sigma_{\mu\nu}+A\!\!\!/A\!\!\!/)^{-1}A\!\!\!/\hat{p}_-\\&\quad+(1-A\!\!\!/)(1-F_{\mu\nu}\sigma_{\mu\nu}+A\!\!\!/A\!\!\!/)^{-1}\hat{p}_+\end{aligned} \tag{3.7}$$

$$\begin{aligned}(1-F_{\mu\nu}\sigma_{\mu\nu}\hat{p}_-+A\!\!\!/\gamma_5)^{-1}&=\hat{p}_++\hat{p}_+A\!\!\!/(1-F_{\mu\nu}\sigma_{\mu\nu}+A\!\!\!/A\!\!\!/)^{-1}(1-A\!\!\!/)\\&\quad+\hat{p}_-(1-F_{\mu\nu}\sigma_{\mu\nu}+A\!\!\!/A\!\!\!/)^{-1}(1-A\!\!\!/)\end{aligned} \tag{3.8}$$

one can expand the first term of (3.6) as

$$\begin{aligned}\operatorname{Tr}\Bigg\{\beta\gamma_5\Bigg[\frac{\mathrm{d}^{2m}}{\mathrm{d}z^{2m}}\Bigg(&\frac{1}{1-F^{(+)}_{\mu\nu}\sigma_{\mu\nu}+z^2A\!\!\!/^{(+)}A\!\!\!/^{(+)}}-z^2A\!\!\!/^{(+)}\frac{1}{1-F^{(+)}_{\mu\nu}\sigma_{\mu\nu}+z^2A\!\!\!/^{(+)}A\!\!\!/^{(+)}}\\&\times A\!\!\!/^{(+)}\frac{1}{1-F^{(+)}_{\mu\nu}\sigma_{\mu\nu}+z^2A\!\!\!/^{(+)}A\!\!\!/^{(+)}}-z^2\frac{1}{1-F^{(+)}_{\mu\nu}\sigma_{\mu\nu}+z^2A\!\!\!/^{(+)}A\!\!\!/^{(+)}}\\&\times A\!\!\!/^{(+)}\frac{1}{1-F^{(+)}_{\mu\nu}\sigma_{\mu\nu}+z^2A\!\!\!/^{(+)}A\!\!\!/^{(+)}}A\!\!\!/^{(+)}\Bigg)\Bigg]_{z=0}\Bigg\}.\end{aligned} \tag{3.9}$$

We recall the integral form of beta function in (2.26)

$$B(m+1,n+1)=\int_0^1 s^n(1-s)^m\,\mathrm{d}s \tag{3.10}$$

which implies that the operation

$$\left.\frac{1}{(2m)!}\frac{\mathrm{d}^{2m}}{\mathrm{d}z^{2m}}\right|_{z=0}$$

in (3.9) is equivalent to the substitution of $(1-s)^m$ for z^{2m}. One can easily derive an expression for the summation of the first term in (2.26) as

$$\begin{aligned}\frac{1}{(4\pi)^n n!}\int_0^1\mathrm{d}s\,\operatorname{Tr}\Bigg[\beta\gamma_5\Bigg(&(s\tilde{F}^{(+)}_{\mu\nu}\sigma_{\mu\nu})^n-s(1-s)\sum_{k=0}^{n-1}A\!\!\!/^{(+)}(s\tilde{F}^{(+)}_{\mu\nu}\sigma_{\mu\nu})^kA\!\!\!/^{(+)}(s\tilde{F}^{(+)}_{\mu\nu}\sigma_{\mu\nu})^{n-k-1}\\&-s(1-s)\sum_{k=0}^{n-1}(s\tilde{F}^{(+)}_{\mu\nu}\sigma_{\mu\nu})^kA\!\!\!/^{(+)}(s\tilde{F}^{(+)}_{\mu\nu}\sigma_{\mu\nu})^{n-k-1}A\!\!\!/^{(+)}\Bigg)\Bigg]\end{aligned} \tag{3.11}$$

where

$$\tilde{F}^{(+)}_{\mu\nu} \equiv F^{(+)}_{\mu\nu} - (1-s)[A^{(+)}_\mu, A^{(+)}_\nu]$$
$$= \partial_\mu A^{(+)}_\nu - \partial_\nu A^{(+)}_\mu + s[A^{(+)}_\mu, A^{(+)}_\nu]. \tag{3.12}$$

Now, we adopt the notation of exterior differential form:

$$A \equiv A_\mu \, dx^\mu \qquad F \equiv \tfrac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu \tag{3.13}$$

$$A_S \equiv sA \qquad F_S \equiv dA_S + A_S^2 \tag{3.14}$$

and recast the expression (3.11) into

$$\frac{1}{(2\pi i)^n n!} \int_0^1 ds \, \mathrm{Tr}\{\beta (F_S^{(+)n} - ns(1-s)[A^{(+)}, P(A^{(+)}, F_S^{(+)n-1})])\} \tag{3.15}$$

or in more compact form

$$\frac{1}{(2\pi i)^n (n-1)!} \int_0^1 ds (1-s) \, \mathrm{Tr}\{\beta \, dP(A^{(+)}, F_S^{(+)n-1})\} \tag{3.16}$$

where $P(\Lambda_1, \Lambda_2, \ldots, \Lambda_n)$ is the symmetrised product, $[A, \omega_P]$ is understood as $[A, \omega_P]_+$ for P odd or $[A, \omega_P]_-$ for P even. With similar treatment to that shown above, one may find that the summation of the second term in (2.26) makes the same contribution. Therefore the final result of the Bardeen anomaly in the case of chiral gauge coupling can be expressed in exterior differential forms as

$$-\delta\Gamma(A^{(+)}, \beta) = 2 \frac{1}{(2\pi i)^n (n-1)!} \int_0^1 ds (1-s) \, \mathrm{Tr}\{\beta \, dP(A^{(+)}, F^{(+)n-1})\}. \tag{3.17}$$

For $2n = 4$, one has

$$\delta\Gamma(A^{(+)}, \beta) = \frac{1}{12\pi^2} \mathrm{Tr}\{\beta \, d(A^{(+)} \, dA^{(+)} + \tfrac{1}{2} A^{(+)3})\}. \tag{3.18}$$

If the Dirac operator takes the form

$$i\not{D} = i\not{\partial} + \tfrac{1}{2}\not{A}^{(-)} \tfrac{1}{2}(1-\gamma_5) \tag{3.19}$$

the result of the Bardeen anomaly becomes

$$-\delta\Gamma(A^{(-)}, \beta) = -2 \frac{1}{(2\pi i)^n (n-1)!} \int_0^1 ds (1-s) \, \mathrm{Tr}\{\beta \, dP(A^{(-)}, F^{(-)n-1})\}. \tag{3.20}$$

3.2. *The Gross–Jackiw anomaly*

The investigation of the Gross–Jackiw anomaly within the path integral scheme may proceed following the same line as described in § 2. Consider the Dirac operator shown as (3.1). By using the notation of Alvarez-Gaumé and Ginsparg (1984, 1985) it can be rewritten as

$$i\not{D} = i\not{\partial}_- + i\not{D}_+ \tag{3.21}$$

where

$$i\not{\partial}_- = i\not{\partial}\tfrac{1}{2}(1-\gamma_5) \qquad i\not{D}_+ = i(\not{\partial} + \not{A}^{(+)})\tfrac{1}{2}(1+\gamma_5). \tag{3.22}$$

Under an infinitesimal gauge transformation

$$A^{(+)} \rightarrow A^{(+)'} = g_+^{-1}(A^{(+)} + d)g_+ \tag{3.23}$$

with

$$g_+ = e^{v_+(x)} \qquad v_+(x) = v_+^a(x)\lambda^a \tag{3.24}$$

where $v_+^a(x)$ are infinitesimal local parameters with λ^a the anti-Hermitian generators of gauge group $G^{(+)}$, the Dirac operator (3.21) transforms as

$$\mathrm{i}\not{D} \rightarrow \mathrm{i}\not{D}(v_+) = \mathrm{i}\not{\partial}_- + g_+^{-1}\mathrm{i}\not{D}_+ g_+ . \tag{3.25}$$

The fact that the classical gauge symmetry is broken at the quamtum level implies

$$\begin{aligned} W(A^{(+)}) &\equiv \int \mathrm{d}\mu(\psi) \exp\left(-\int \bar{\psi}\mathrm{i}\not{D}\psi\right) \\ &\neq W(A^{(+)}; v_+) \\ &\equiv \int \mathrm{d}\mu(\psi) \exp\left(-\int \bar{\psi}\mathrm{i}\not{D}(v_+)\psi\right). \end{aligned} \tag{3.26}$$

We thus define the anomalous factor $\exp(-\delta\Gamma)$ induced by (3.24) as the ratio of two fermion determinants

$$\exp(-\delta\Gamma(v_+)) \equiv \frac{W(A^{(+)}, v_+)}{W(A^{(+)})} = \frac{\det \mathrm{i}\not{D}(v_+)}{\det \mathrm{i}\not{D}} . \tag{3.27}$$

$\delta\Gamma(v)$ can be found by treating $\delta\mathrm{i}\not{D}$ as a perturbation in the diagonalising representation of $\mathrm{i}\not{D}$, here

$$\begin{aligned} \delta\mathrm{i}\not{D} &\equiv \mathrm{i}\not{D}(v_+) - \mathrm{i}\not{D} \\ &= g_+^{-1}\mathrm{i}\not{D}_+ g_+ - \mathrm{i}\not{D}_+ \\ &= [\mathrm{i}\not{D}_+, v_+]. \end{aligned} \tag{3.28}$$

Let $\{\phi_n(x)\}$, $\{\tilde{\phi}_n(x)\}$ be the complete sets of two Hermitian operators, i.e.

$$\begin{aligned} [(\mathrm{i}\not{D})^\dagger(\mathrm{i}\not{D})]\phi_n(x) &= \lambda_n^2\phi_n(x) \\ [(\mathrm{i}\not{D})(\mathrm{i}\not{D})^\dagger]\tilde{\phi}_n(x) &= \lambda_n^2\tilde{\phi}_n(x) \end{aligned} \tag{3.29}$$

with mapping relations for $\lambda_n \neq 0$:

$$\begin{aligned} (\mathrm{i}\not{D})\phi_n(x) &= \lambda_n \exp(\mathrm{i}\theta_n)\tilde{\phi}_n(x) \\ (\mathrm{i}\not{D})^\dagger\tilde{\phi}_n(x) &= \lambda_n \exp(-\mathrm{i}\theta_n)\phi_n(x) \end{aligned} \tag{3.30}$$

and diagonalising expression

$$\int \tilde{\phi}_n^\dagger(x)\mathrm{i}\not{D}\phi_n(x) = \lambda_n \exp(\mathrm{i}\theta_n). \tag{3.31}$$

Using the operators $\hat{p}_\pm \equiv \frac{1}{2}(1 \pm \gamma_5)$, one easily finds the chiral projective mapping relations from (3.30):

$$\begin{aligned} (\mathrm{i}\not{D}_+)\phi_n^{(+)}(x) &= \lambda_n \exp(\mathrm{i}\theta_n)\tilde{\phi}_n^{(-)}(x) \\ (\mathrm{i}\not{D}_+)^\dagger\tilde{\phi}_n^{(-)}(x) &= \lambda_n \exp(-\mathrm{i}\theta_n)\phi_n^{(+)}(x) \end{aligned} \tag{3.32}$$

where

$$\phi_n^{(\pm)}(x)=\tfrac{1}{2}(1\pm\gamma_5)\phi_n(x)$$
$$\tilde{\phi}_n^{(\pm)}(x)=\tfrac{1}{2}(1\pm\gamma_5)\tilde{\phi}_n(x). \tag{3.33}$$

Hence, to the first order of $v^a(x)$, the correction to any diagonal element of $\mathrm{i}\not{D}$ due to $\delta\mathrm{i}\not{D}$ shown in (3.28) can be derived with the help of (3.32) as

$$\begin{aligned}\int \mathrm{d}x\,\tilde{\phi}_n^{\dagger}(x)\delta\mathrm{i}\not{D}\phi_n(x)\\ &=\int \mathrm{d}x\,\tilde{\phi}_n^{\dagger}(x)[\mathrm{i}\not{D}_+, v_+]\phi_n(x)\\ &=\int \mathrm{d}x\,\tilde{\phi}_n^{(-)\dagger}(x)(\mathrm{i}\not{D}_+v_+)\phi_n^{(+)}(x)-\int \mathrm{d}x\,\tilde{\phi}_n^{(-)\dagger}(x)(v_+\mathrm{i}\not{D}_+)\phi_n^{(+)}(x)\\ &=\lambda_n\exp(\mathrm{i}\theta_n)\int \mathrm{d}x[\phi_n^{(+)\dagger}(x)v_+\phi_n^{(+)}(x)-\tilde{\phi}_n^{(-)\dagger}(x)v_+\tilde{\phi}_n^{(-)}(x)]\end{aligned} \tag{3.34}$$

and the diagonal element of the Dirac operator (3.25) acquires a phase factor $\exp(\mathrm{i}\delta\theta_n)$ with

$$\begin{aligned}\mathrm{i}\delta\theta_n&=\int \mathrm{d}x(\phi_n^{(+)\dagger}v_+\phi_n^{(+)}-\tilde{\phi}_n^{(-)\dagger}v_+\tilde{\phi}_n^{(-)})\\ &=\int \mathrm{d}x[\phi_n^{\dagger}v_+\tfrac{1}{2}(1+\gamma_5)\phi_n-\tilde{\phi}_n^{\dagger}v_+\tfrac{1}{2}(1-\gamma_5)\tilde{\phi}_n].\end{aligned} \tag{3.35}$$

It should be pointed out that the above expression is also correct even for the zero diagonal elements. This can be easily proved by adding a term '$\varepsilon\gamma_5$' into $\mathrm{i}\not{D}$ with the limit $\varepsilon\to 0$ as mentioned in I. We thus find

$$\begin{aligned}-\delta\Gamma(v_+)&=\sum_n \mathrm{i}\delta\theta_n\\ &=\int \mathrm{d}x\left\{\sum_n \phi_n^{\dagger}v_+\tfrac{1}{2}(1+\gamma_5)\phi_n-\sum_n \tilde{\phi}_n^{\dagger}v_+\tfrac{1}{2}(1-\gamma_5)\tilde{\phi}_n\right\}.\end{aligned} \tag{3.36}$$

Since both $\{\phi_n\}$ and $\{\tilde{\phi}_n\}$ are complete sets, the terms $(\mathrm{Tr}\,v_+-\mathrm{Tr}\,v_+)$ appearing in (3.36) cancel each other and giving

$$-\delta\Gamma(v_+)=\tfrac{1}{2}\int \mathrm{d}x\left[\sum_n \phi_n^{\dagger}v_+\gamma_5\phi_n+\sum_n \tilde{\phi}_n^{\dagger}v_+\gamma_5\tilde{\phi}_n\right]. \tag{3.37}$$

Under the null space regularisation scheme as argued in I, the regularised $\delta\Gamma(v_+)$ can be expressed as

$$-\delta\Gamma(v_+)=\tfrac{1}{2}[\mathrm{Tr}\,v_+\gamma_5 f(\xi,\mathrm{i}\not{D})]_{\xi\text{ independent}}+\tfrac{1}{2}[\mathrm{Tr}\,v_+\gamma_5 f(\xi,(\mathrm{i}\not{D})^{\dagger})]_{\xi\text{ independent}}. \tag{3.38}$$

Here $\mathrm{i}\not{D}$ is just the expression of (3.1). Therefore, the Gross-Jackiw anomaly under the infinitesimal gauge transformation (3.24) is similar to that of the Bardeen anomaly shown as (3.17) with only one difference of factor $\frac{1}{2}$. An unified formula of Gross-Jackiw anomalies for both operators (3.1) and (3.19) are thus derived within the path integral scheme as

$$-\delta\Gamma(v_\pm)=\pm\frac{1}{(2\pi\mathrm{i})^n(n-1)!}\int_0^1 \mathrm{d}s(1-s)\,\mathrm{Tr}\{v_\pm\,\mathrm{d}P(A^{(\pm)},F_S^{(\pm)n-1})\} \tag{3.39}$$

where $g_\pm=\mathrm{e}^{v_\pm}\in G^{(\pm)}$ and $G^{(\pm)}$ is the gauge group related to $A^{(\pm)}$.

In the case of gauge transformation along a path in the group space with the gauge group element $g_{\pm}(x, \tau) \in G^{(\pm)}$ specified by variables x_μ in spacetime and τ_i in group space, we have two exterior differentiations (Zumino 1985)

$$d \equiv \mathrm{d}x_\mu \frac{\partial}{\partial x_\mu} \qquad \delta \equiv \mathrm{d}\tau_i \frac{\partial}{\partial \tau_i} \tag{3.40}$$

which satisfy

$$d^2 = \delta^2 = d\delta + \delta d = 0. \tag{3.41}$$

Denoting

$$A_g = g^{-1}(A + d)g \qquad v = g^{-1}\delta g \tag{3.42}$$

one can express the anomalous factor of generating function induced by any piece of infinitesimal gauge transformation $\tau_i \to \tau_i + \mathrm{d}\tau_i$ by

$$\mathrm{e}^{-\delta\Gamma(A_g)} = \frac{W(A_g, v)}{W(A_g)} = \frac{\det \mathrm{i}\not{D}(A_g, v)}{\det \mathrm{i}\not{D}(A_g)}. \tag{3.43}$$

So, the result of $\delta\Gamma(A_g)$ is simply equation (3.39) with $A^{(\pm)}$ understood as $A_{g_\pm}^{(\pm)}$ and $v_\pm$ as $g_\pm^{-1}\delta g_\pm$.

4. The topological invariant and ω_{2n+1} forms for non-Abelian anomalies

Furthermore, one can trace out all the differential geometric objects based on formula (3.39) which we derived in the path integral formalism. One will also see in § 5 that these objects are very useful for applying differential geometric techniques to the path integral derivation of anomalous effective action induced by finite chiral transformation.

4.1. *The topological invariants*

After defining

$$\omega_{2n}^1(A^{(\pm)}) = \pm(n+1)n \int_0^1 s(1-s)\,\mathrm{tr}\{v_\pm \,\mathrm{d}P(A^{(\pm)}, F_S^{(\pm)n-1})\} \tag{4.1}$$

$$K_{n+1} = \frac{1}{(2\pi \mathrm{i})^{n+1}(n+1)!} \tag{4.2}$$

(3.39) becomes

$$\frac{1}{2\pi\mathrm{i}}\,\delta\Gamma(A^{(\pm)}) = K_{n+1}\int_{S^{2n}} \omega_{2n}^1(A^{(\pm)}) \tag{4.3}$$

where S^{2n} is a compactified $2n$-dimensional Euclidean space. Let D^{2n+1} be a $(2n+1)$-dimensional disc whose boundary is $\partial D^{2n+1} = S^{2n}$. One can rewrite (4.3) via Stokes' theorem as

$$\int_{S^{2n}} \omega_{2n}^1(A^{(\pm)}) = \int_{D^{2n+1}} \mathrm{d}\omega_{2n}^1(A^{(\pm)}) = \int_{D^{2n+1}} D_S\omega_{2n}^1(A^{(\pm)}) \tag{4.4}$$

where

$$D_S(*) \equiv d(*) + [A_S, (*)]. \tag{4.5}$$

The last step in (4.4) follows from the fact that the result of trace is a singlet. It is easy to verify the Bianchi identities

$$D_S F_S^{(\pm)} = 0 \tag{4.6}$$

and

$$D_S A = \partial F_S / \partial S. \tag{4.7}$$

Defining again Lie algebra valued 1-forms in group space:

$$v_\pm = g_\pm^{-1} \delta g_\pm \qquad g_\pm \in \mathrm{G}^{(\pm)} \tag{4.8}$$

it is also easy to verify the BRS transformation (Zumino 1985)

$$\begin{aligned} \delta v_\pm &= -v_\pm^2 \\ \delta A^{(\pm)} &= -\mathrm{d} v_\pm - [v_\pm, A^{(\pm)}] = -D v_\pm . \end{aligned} \tag{4.9}$$

Here $A^{(\pm)}$ should be understood as $A_{g_\pm}^{(\pm)}$, D as the covariant differentiations. With the aid of (4.6)-(4.9), one can show that

$$D_S \omega_{2n}^1 (A^{(\pm)}) = -\delta \omega_{2n+1}(A^{(\pm)}) \tag{4.10}$$

where

$$\omega_{2n+1}(A^{(\pm)}) \equiv \pm (n+1) \int_0^1 \mathrm{d}s \ \mathrm{str}(A^{(\pm)}, F_S^{(\pm) n}) \tag{4.11}$$

in which str is a symmetrised trace. Substituting (4.10), (4.11) and (4.4) into (4.3), one can write

$$\frac{1}{2\pi \mathrm{i}} \delta \Gamma(A^{(\pm)}) = -\int_{D^{2n+1}} \delta \Lambda_{2n+1}(A^{(\pm)}) \tag{4.12}$$

where $\Lambda_{2n+1}(A^{(\pm)})$ is defined as

$$\Lambda_{2n+1}(A^{(\pm)}) = K_{n+1} \omega_{2n+1}(A^{(\pm)}) \tag{4.13}$$

with K_{n+1} and $\omega_{2n+1}(A^{(\pm)})$ shown in (4.2) and (4.11). By observing

$$\mathrm{d}\omega_{2n+1}(A^{(\pm)}) = \mathrm{tr}\, F^{(\pm) n+1}$$

one finds

$$\begin{aligned} \mathrm{d}\Lambda_{2n+1}(A^{(\pm)}) &= K_{n+1} \,\mathrm{tr}\, F^{(\pm) n+1} \\ &= \frac{1}{(2\pi \mathrm{i})^{n+1} (n+1)!} \mathrm{tr}\, F^{(\pm) n+1} \end{aligned} \tag{4.14}$$

which is just the $(2n+2)$-dimensional Atiyah–Singer index density (Atiyah and Singer 1968). The detailed topological meaning of the $2n$-dimensional non-Abelian anomaly and the $(2n+2)$-dimensional index density has been discussed by Alvarez-Gaumé and Ginsparg (1984). Here we derive this relation substantially within the path integral scheme.

For the more general Dirac operator shown in (2.2), one can rewrite it via chiral projection as

$$\mathrm{i}\not{D} = \mathrm{i}(\not{\partial} + A^{(+)})\tfrac{1}{2}(1+\gamma_5) + \mathrm{i}(\not{\partial} + A^{(-)})\tfrac{1}{2}(1-\gamma_5) \tag{4.15}$$

with

$$A^{(\pm)} = V \pm A. \tag{4.16}$$

So, the combined topological invariant may be expressed as

$$\mathrm{d}\Lambda^{\mathrm{L-R}}_{2n+1}(A^{(+)}, A^{(-)}) = K_{n+1}(\mathrm{tr}\, F^{(+)n+1} - \mathrm{tr}\, F^{(-)n+1}) \tag{4.17}$$

or expressed in $V-A$ form:

$$\mathrm{d}\Lambda^{\mathrm{L-R}}_{2n+1} = \frac{K_{n+1}}{2^{n-1}}\,\mathrm{tr}[\gamma_5(F^{(V)} + F^{(A)}\gamma_5)^{n+1}]. \tag{4.18}$$

Here the Dirac matrices are taken to be $2^n \times 2^n$ dimensions.

4.2. *The* $(2n+1)$*-forms for chiral anomaly*

From (4.13) we see that the $(2n+1)$ forms for gauge anomalies of the Dirac operator (4.15) in left-right formalism are simply

$$\Lambda^{\mathrm{L-R}}_{2n+1}(A^{(+)}, A^{(-)}) = K_{n+1}[\omega_{2n+1}(A^{(+)}) + \omega_{2n+1}(A^{(-)})] \tag{4.19}$$

where K_{n+1} and ω_{2n+1} are shown in (4.2) and (4.11). As expressed in (4.12) and (4.18), $\Lambda^{\mathrm{L-R}}_{2n+1}$ is related both to the local index density and to the gauge anomaly. It is of great importance in the approach of the mathematical structure of anomalies. Now, we are in a position to derive an expression for $(2n+1)$-forms in $V-A$ formalism denoted by $\Lambda^{V-A}_{2n+1}(V, A)$. Noting that the conservation of vector current in the $V-A$ scheme implies the invariance of Λ^{V-A}_{2n+1} under the infinitestimal vector transformation. What we try to seek is a solution of the differential equations (see (4.18))

$$\mathrm{d}\Lambda^{V-A}_{2n+1}(V, A) = \frac{K_{n+1}}{2^{n-1}}\,\mathrm{tr}[\gamma_5(F^{(V)} + F^{(A)}\gamma_5)^{n+1}] \tag{4.20}$$

under the constraint

$$\delta_\alpha \Lambda^{V-A}_{2n+1}(V, A) = 0. \tag{4.21}$$

Here $\alpha = \alpha^a(x)\lambda^a$ specifies a vector transformation element $g = \mathrm{e}^{\alpha(x)}$.

Since, for infinitesimal $\alpha^a(x)$ one always has

$$\mathrm{e}^{\alpha(x)} = g_+\tfrac{1}{2}(1+\gamma_5) + g_-\tfrac{1}{2}(1-\gamma_5) \tag{4.22}$$

with

$$g_\pm = \mathrm{e}^{v_\pm} \qquad v_\pm = \alpha(x) \tag{4.23}$$

the constraint (4.21) can be rewritten as

$$\delta_{v_+}\Lambda^{V-A}_{2n+1}(A^{(+)} + A^{(-)}, A^{(+)} - A^{(-)} + \delta_{v_-}\Lambda^{V-A}_{2n+1}(A^{(+)} + A^{(-)}, A^{(+)} - A^{(-)}) = 0. \tag{4.24}$$

One can derive Λ^{V-A}_{2n+1} by introducing two parameters S_+, S_- in defining a $(2n+2)$-form as

$$\Lambda_{2n+2}(S_+, S_-) = \frac{K_{n+1}}{2^{n+1}}\,\mathrm{tr}(\gamma_5 F_Z^{n+1}) \tag{4.25}$$

with

$$F_Z(S_+, S_-) \equiv \mathrm{d}Z(S_+, S_-) + Z^2(S_+, S_-) \tag{4.26}$$

$$Z(S_+, S_-) \equiv S_+ W_+ + S_- W_- \tag{4.27}$$

where

$$W_\pm = V \pm A\gamma_5 = A^{(+)}\tfrac{1}{2}(1 \pm \gamma_5) + A^{(-)}\tfrac{1}{2}(1 \mp \gamma_5). \tag{4.28}$$

Using the following equalities:

$$D_Z F_Z = 0 \tag{4.29}$$

$$D_Z W_\pm = \partial F_Z / \partial S_\pm \tag{4.30}$$

with D_Z defined as

$$D_Z(*) = \mathrm{d}(*) + [Z, (*)] \tag{4.31}$$

one arrives at

$$\delta_S \Lambda_{2n+2}(S_+, S_-) = K_{n+1} \frac{(n+1)}{2^{n-1}}\, d\{\mathrm{str}[\gamma_5(\delta_S Z, F_Z^n)]\} \tag{4.32}$$

or, more explicitly,

$$\delta_S \Lambda_{2n+1}(S_+, S_-) = K_{n+1} \frac{(n+1)}{2^{n-1}}\, \mathrm{str}[\gamma_5(\delta_S Z, F_Z^n)]. \tag{4.33}$$

Here δ_S is understood as ordinary variations of parameters $S_\pm$. It is easy to check that the constraint demands

$$S_+ + S_- = 1. \tag{4.34}$$

Since

$$\begin{aligned}\Lambda_{2n+2}(S_+ = 1, S_- = 0) &= \Lambda_{2n+2}(S_+ = 0, S_- = 1)\\ &= K_{n+1} \frac{1}{2^{n-1}}\, \mathrm{tr}[\gamma_5(F^{(V)} + F^{(A)}\gamma_5)^{n+1}]\end{aligned} \tag{4.35}$$

one finds the integration must be carried out along a straight line from point $P(1, 0)$ to point $Q(0, 1)$ on the (S_+, S_-) plane. One obtains

$$\Lambda_{2n+1}^{V-A}(A^{(+)} + A^{(-)}, A^{(+)} - A^{(-)}) = K_{n+1} \frac{(n+1)}{2^n} \int_0^1 \mathrm{str}[\gamma_5(\delta_S Z, F_Z^n)] \tag{4.36}$$

in which

$$Z = [sA^{(+)} + (1-s)A^{(-)}]\tfrac{1}{2}(1+\gamma_5) + [sA^{(-)} + (1-s)A^{(+)}]\tfrac{1}{2}(1-\gamma_5) \tag{4.37}$$

$$F_Z = \mathrm{d}Z + Z^2. \tag{4.38}$$

Returning to variables (V, A), one has

$$\Lambda_{2n+1}^{V-A}(V, A) = K_{n+1} \frac{(n+1)}{2^{n-1}} \int_0^1 \mathrm{d}s\, \mathrm{str}\{A, [F^{(V)} + (2s-1)\gamma_5 F^{(A)} - 4s(s-1)A^2]^n\}. \tag{4.39}$$

As an example for deriving the chiral anomaly from Λ_{2n+1}^{V-A}, consider the case of $2n=4$, one easily finds from (4.39)

$$\Lambda_5^{V-A}(V,A)=\frac{1}{12\pi^2}\operatorname{tr}(3AF^{(V)^2}+AF^{(A)^2}-4A^3F^{(V)}+\tfrac{8}{5}A^5). \tag{4.40}$$

According to the definition for infinitesimal chiral transformation

$$\delta_\beta(V+A\gamma_5)=\mathrm{e}^{-\beta\gamma_5}(V+A\gamma_5+d)\,\mathrm{e}^{\beta\gamma_5}-(V+A\gamma_5) \tag{4.41}$$

one has

$$\begin{aligned}&\delta_\beta V=[A,\beta] &&\delta_\beta A=D_V\beta\\ &\delta_\beta F^{(V)}=[F^{(A)},\beta] &&\delta_\beta F^{(A)}=[F^{(V)},\beta]\end{aligned} \tag{4.42}$$

and

$$\begin{aligned}&D_VV=F^{(V)}+V^2-A^2 &&D_VA=F^{(A)}\\ &D_VF^{(V)}=[F^{(A)},A] &&D_VF^{(A)}=[F^{(V)},A]\end{aligned} \tag{4.43}$$

where

$$D_V\equiv d(*)+[V,(*)]. \tag{4.44}$$

The chiral anomaly can thus be obtained via the following relation:

$$\delta_\beta\Lambda_{2n+1}^{V-A}(V,A)=-\mathrm{d}\Lambda_{2n}^1(V,A). \tag{4.45}$$

For $2n=4$, one gets

$$\begin{aligned}\delta\Gamma(V,A)&=2\pi\mathrm{i}\int_{S^4}\Lambda_4^1(V,A)\\ &=\frac{1}{12\pi^2}\operatorname{tr}\{\beta[F^{(A)^2}+3F^{(V)^2}-4(A^2F^{(V)}+AF^{(V)}A+F^{(V)}A^2)+8A^4]\}\end{aligned} \tag{4.46}$$

which is just the Bardeen anomaly (2.30) in exterior differential forms (for example, Kawai and Tye 1984, Ingermanson 1985).

4.3. *Bardeen counterterms*

Since both exterior derivatives of $\Lambda_{2n+1}^{\mathrm{L-R}}$ (4.18) and Λ_{2n+1}^{V-A} (4.20) are the same:

$$\mathrm{d}\Lambda_{2n+1}^{\mathrm{L-R}}=\mathrm{d}\Lambda_{2n+1}^{V-A}=K_{n+1}\frac{1}{2^{n-1}}\operatorname{tr}[\gamma_5(F^{(V)}+F^{(A)}\gamma_5)^{n+1}] \tag{4.47}$$

one has

$$\Lambda_{2n+1}^{\mathrm{L-R}}-\Lambda_{2n+1}^{V-A}=\mathrm{d}\rho_{2n} \tag{4.48}$$

and express it as an integral along the closed triangle path on the (S_+,S_-) plane:

$$\mathrm{d}\rho_{2n}=K_{n+1}\frac{(n+1)}{2^n}\oint_{\Delta OPQ}\operatorname{str}\{\gamma_5(\delta_SZ,F_Z^n)\} \tag{4.49}$$

with the points $O(S_+=0,S_-=0)$, $P(1,0)$ and $Q(0,1)$. Using (4.26)-(4.30), one derives

$$\mathrm{d}\rho_{2n}=K_{n+1}\frac{n(n+1)}{2^n}d\int_0^1\mathrm{d}S_+\int_0^{1-S_+}\mathrm{d}S_-\operatorname{str}\{\gamma_5W_+W_-F_Z^{n-1}\}. \tag{4.50}$$

As an example, for $2n=4$, one easily finds the Bardeen counterterms $\rho_4(A^{(+)}, A^{(-)})$ as

$$\rho_4=\frac{1}{48\pi^2}\operatorname{tr}[(A^{(+)}A^{(-)}-A^{(-)}A^{(+)})(F^{(+)}+F^{(-)}) \\ -(A^{(+)3}A^{(-)}-A^{(-)3}A^{(+)})+\tfrac{1}{2}A^{(+)}A^{(-)}A^{(+)}A^{(-)}] \tag{4.51}$$

(Bardeen 1969, Kawai and Tye 1984, Mañes 1985).

5. The anomalous effective action induced by finite chiral transformation

As pointed out in I, the reformulated path integral scheme is especially suitable for calculating the anomalous effective action induced by finite chiral rotation. Consider the following transformation:

$$\psi_0\to\psi_1=\exp(\xi(x)\gamma_5)\psi_0 \qquad \bar\psi_0\to\bar\psi_1=\bar\psi_0\exp(\xi(x)\gamma_5) \tag{5.1}$$

with

$$g=\exp(\xi(x)\gamma_5)\in \mathrm{G}\times\mathrm{G} \qquad \xi(x)=\lambda^a\xi^a(x). \tag{5.2}$$

λ^a are anti-Hermitian generators of the group, $\xi^a(x)$ are finite scalar fields. For example, $\xi^a(x)\lambda^a F_\pi$ with $F_\pi=93$ MeV may be understood as $\pi^a\lambda^a$, π^a being pion fields in QCD. In our treatment, a parameter t is introduced with interval 0-1, that is to define

$$\psi_t\equiv \mathrm{e}^{t\xi(x)}\psi_0 \qquad \bar\psi_t\equiv\bar\psi_0\,\mathrm{e}^{t\xi(x)\gamma_5} \tag{5.3}$$

and to split the range 0-1 into N equal infinitesimal intervals $\mathrm{d}t(N\to\infty)$, the path integral method for successive chiral transformation described in § 2 can thus be applied. One can write the anomalous factor induced by any infinitesimal section of chiral transformation specified by $t\to t+\mathrm{d}t$ as

$$\exp\{-\delta_t\Gamma(V_t, A_t)\}=\frac{W(V_t, A_t;\mathrm{d}t\,\xi(x))}{W(V_t, A_t)}=\frac{\det \mathrm{i}\not{D}(V_t, A_t;\mathrm{d}t\,\xi(x))}{\det \mathrm{i}\not{D}(V_t, A_t)} \tag{5.4}$$

where

$$\mathrm{i}\not{D}(V_t, A_t)\equiv \mathrm{i}(\not{\partial}+\not{V}_t+\not{A}_t) \tag{5.5}$$

$$\mathrm{i}\not{D}(V_t, A_t;\mathrm{d}t\,\xi(x))\equiv\exp(\xi(x)\,\mathrm{d}t)\mathrm{i}\not{D}(V_t, A_t)\exp(\xi(x)\,\mathrm{d}t) \tag{5.6}$$

with V_t, A_t defined according to

$$V_t+A_t\gamma_5\equiv\exp(t\xi(x)\gamma_5)(V+A\gamma_5+d)\exp(t\xi(x)\gamma_5). \tag{5.7}$$

In the diagonalised representation of Dirac operator (5.5) which is comoving with parameter t, one can treat $\delta \mathrm{i}\not{D}(V_t, A_t)$ as a perturbation

$$\delta\mathrm{i}\not{D}(V_t, A_t)\equiv \mathrm{i}\not{D}(V_t, A_t;\mathrm{d}t\,\xi(x))-\mathrm{i}\not{D}(V_t, A_t) \\ =\{\mathrm{i}\not{D}(V_t, A_t), \xi(x)\,\mathrm{d}t\}. \tag{5.8}$$

By comparing it with (2.20), one finds $\delta_t\Gamma(V_t, A_t)$ in (5.4) can be obtained simply by replacing β, V, A in (2.26)-(2.28) with $\mathrm{d}t\,\xi(x)$, V_t, A_t respectively, i.e.

$$-\delta_t\Gamma(V_t, A_t)=\frac{1}{(4\pi)^n n!}\operatorname{Tr}\Bigg\{\mathrm{d}t\,\xi(x)\gamma_5\sum_{m=0}^{n}B(m+1, n+1) \\ \times\left[\frac{1}{(2m)!}\frac{\mathrm{d}^{2m}}{\mathrm{d}z^{2m}}\sum_{l_1+l_2=m+n}(Q_+^{l_1}(t)Q_-^{l_2}(t)+Q_-^{l_1}(t)Q_+^{l_2}(t))\right]_{z=0}\Bigg\} \tag{5.9}$$

where

$$Q_{\pm}(t)=(F_t^{(V)})_{\mu\nu}\sigma_{\mu\nu}\pm(F_t^{(A)})_{\mu\nu}\sigma_{\mu\nu}\gamma_5+2z\not{A}\gamma_5 \tag{5.10}$$

$$F_t^{(V)}=\mathrm{d}V_t+V_t^2+A_t^2 \qquad F_t^{(A)}=\mathrm{d}A_t+\{V_t,A_t\}. \tag{5.11}$$

For example, $2n=4$, the Bardeen anomaly in differential forms shown in (4.46) should be rewritten as

$$\delta_t\Gamma(V_t,A_t)=2\pi\mathrm{i}\int_{S^4}\Lambda_4^1(V_t,A_t) \tag{5.12}$$

with

$$2\pi\mathrm{i}\Lambda_4^1(V_t,A_t)=\frac{1}{12\pi^2}\operatorname{tr}\{\mathrm{d}t\,\xi[F_t^{(A)2}+3F_t^{(V)2}-4(A^2F_t^{(V)}+A_tF_t^{(V)}A_t+F_t^{(V)}A_t^2)+8A_t^4]\}. \tag{5.13}$$

In principle, the anomalous effective action Γ^{WZ} which is referred to as the Wess-Zumino terms in some literature (Witten 1983, Ingermanson 1985, Mañes 1985) can be obtained by integrating over t. But, actually, the t dependence of A_t, V_t, $F_t^{(A)}$, $F_t^{(V)}$ shown in (5.7) and (5.11) leads to complications. However, similar to (4.12), one can turn (5.9) into an integral over a disc D^{2n+1}, i.e.

$$\begin{aligned}\delta_t\Gamma(V_t,A_t)&=2\pi\mathrm{i}\int_{S^{2n}}\Lambda_{2n}^1(V_t,A_t)\\&=2\pi\mathrm{i}\int_{D^{2n+1}}\mathrm{d}\Lambda_{2n}^1(V_t,A_t)\\&=-2\pi\mathrm{i}\int_{D^{2n+1}}\delta_{(\mathrm{d}t\xi)}\Lambda_{2n+1}^{V-A}(V_t,A_t)\end{aligned} \tag{5.14}$$

where $\Lambda_{2n+1}^{V-A}(V_t,A_t)$ is shown in (4.39) with $\beta, V, A, F^{(V)}, F^{(A)}$ replaced by $\mathrm{d}t\,\xi$, V_t, A_t, $F_t^{(V)}$, $F_t^{(A)}$ respectively. Observing that

$$\begin{aligned}\delta_{(\xi\,\mathrm{d}t)}(V_t+\gamma_5A_t)&=\exp(-\mathrm{d}t\,\xi\gamma_5)(V_t+A_t\gamma_5+d)\exp(\mathrm{d}t\,\xi\gamma_5)-(V_t+A_t\gamma_5)\\&=[(V_t+A_t\gamma_5+d),\xi\gamma_5]\,\mathrm{d}t\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}\delta_t(V_t+\gamma_5A_t)&=\frac{\partial}{\partial t}[\exp(-t\xi\gamma_5)(V+A\gamma_5+d)\exp(t\xi\gamma_5)]\,\mathrm{d}t\\&=[(V_t+A_t\gamma_5+d),\xi\gamma_5]\,\mathrm{d}t\end{aligned} \tag{5.16}$$

one finds $\delta_{(\mathrm{d}t\xi)}$ in (5.14) is equivalent to δ_t and can rewrite (5.14) as

$$-\delta_t\Gamma(V_t,A_t)=2\pi\mathrm{i}\int_{D^{2n+1}}\delta_t\Lambda_{2n+1}^{V-A}(V_t,A_t). \tag{5.17}$$

Now the integration over t becomes very simple. It gives

$$-\Gamma^{\mathrm{WZ}}=2\pi\mathrm{i}\int_{D^{2n+1}}[\Lambda_{2n+1}^{V-A}(V_{t=1},A_{t=1})-\Lambda_{2n+1}^{V-A}(V_{t=0},A_{t=0})] \tag{5.18}$$

and the differential geometric technique can be easily applied in this calculation. First, one can write

$$\Lambda_{2n+1}^{V-A}(V_{t=1}, A_{t=1}) - \Lambda_{2n+1}^{V-A}(V_{t=0}, A_{t=0}) = \Lambda_{2n+1}^{V-A}(g^{-1}\,\mathrm{d}g, 0) + \mathrm{d}R_{2n}(V_{t=0}, A_{t=0}, g) \tag{5.19}$$

in which $g = \exp(\xi(x)\gamma_5)$ as shown in (5.2). This leads to

$$\Gamma^{\mathrm{WZ}} = \Gamma_{2n+1}^{\mathrm{WZ}} + \Gamma_{2n}^{\mathrm{WZ}} \tag{5.20}$$

where

$$\Gamma_{2n+1}^{\mathrm{WZ}} = 2\pi\mathrm{i}\int_{D^{2n+1}} \Lambda_{2n+1}^{V-A}(g^{-1}\,\mathrm{d}g, 0) \tag{5.21}$$

$$\Gamma_{2n}^{\mathrm{WZ}} = 2\pi\mathrm{i}\int_{S^{2n}} R_{2n}(V_{t=0}, A_{t=0}, g). \tag{5.22}$$

$\Gamma_{2n+1}^{\mathrm{WZ}}$ can be read off from (4.39). The explicit expression is

$$\Gamma_{2n+1}^{\mathrm{WZ}} = -2\frac{\mathrm{i}^n}{(4\pi\mathrm{i})^n n!} B(n+1, n+1)\int_{D^{2n+1}} \mathrm{tr}[\gamma_5(g^{-1}\,\mathrm{d}g)^{2n+1}]. \tag{5.23}$$

$\Gamma_{2n}^{\mathrm{WZ}}$ can be derived by applying Cartan's homotopy operator $\hat{k}_t$ (see, for example, Zumino 1983, Song 1986) the result is

$$-\Gamma_{2n}^{\mathrm{WZ}} = 2\pi\mathrm{i}\int_{S^{2n}} \hat{k}_t\Lambda_{2n+1}^{V-A}(V_t, A_t)\bigg|_{t=0}^{t=1} + 2\pi\mathrm{i}\int_{S^{2n}} \hat{k}_0\hat{k}_1\Lambda_{2n+1}^{V-A}(V_{t=1}, A_{t=1}). \tag{5.24}$$

Here $\hat{k}_t$ is defined by

$$\hat{k}_t\mathcal{P}(V_t, A_t, F_t^{(V)}, F_t^{(A)}) = \int_0^1 k_t\mathcal{P}(V_t^s, A_t^s, F_t^{(V)s}, F_t^{(A)s}) \tag{5.25}$$

with

$$\begin{aligned} &k_t V_t^s = 0 \qquad && k_t A_t^s = 0 \\ &k_t F_t^{(V)s} = \mathrm{d}s\, V_t \qquad && k_t F_t^{(A)s} = \mathrm{d}s\, A_t \end{aligned} \tag{5.26}$$

and the comoving interpolating fields

$$\begin{aligned} &V_t^s = sV_t \qquad A_t^s = sA_t \\ &F_t^{(V)s} = sF_t^{(V)} - s(1-s)(V_t^2 + A_t^2) \\ &F_t^{(A)s} = sF_t^{(A)} - s(1-s)[V_t, A_t]. \end{aligned} \tag{5.27}$$

One thus generalises the gauge Wess-Zumino effective action in $V-A$ formalism presented by Ingermanson (1985) to $2n$-dimensional space. For $2n = 4$, one has

$$\begin{aligned} -\Gamma^{\mathrm{WZ}} = {} & \frac{1}{24\pi^2}\int_{S^4} \mathrm{tr}[2A_t\{V_t, F_t^{(V)}\} + V_t^3 A_t + 3V_t A_t^3]\big|_{t=0}^{t=1} \\ & + \frac{1}{96\pi^2}\int_{S^4} \mathrm{tr}\{\gamma_5[\mathrm{d}g\,g^{-1}\{W_0, F_{W_0}\} - \mathrm{d}g\,g^{-1}W_0^3 + \tfrac{1}{2}(W_0\,\mathrm{d}g\,g^{-1})^2 \\ & + W_0(\mathrm{d}g\,g^{-1})^3]\} - \frac{1}{480\pi^2}\int_{D^5} \mathrm{tr}\{\gamma_5(g^{-1}\,\mathrm{d}g)^5\} \end{aligned} \tag{5.28}$$

with

$$W_0 \equiv V_0 + A_0\gamma_5 \qquad F_{W_0} \equiv \mathrm{d}W_0 + W_0^2. \tag{5.29}$$

The anomalous action induced by finite chiral transformation (5.1) can also be turned into L-R formalism, but the Bardeen counterterms must be taken into consideration carefully. The final result of the gauged Wess-Zumino term can be expressed in $A^{(+)}, A^{(-)}, u = \mathrm{e}^{2\xi(x)}$ as

$$\Gamma^{\mathrm{WZ}}(A^{(+)}, A^{(-)}) = 2\pi\mathrm{i}\int_{S^{2n}} \rho_{2n}(A^{(+)}, A^{(-)}) + 2\pi\mathrm{i}\int_{S^{2n}} \tilde{\rho}_{2n}(A^{(+)}, u)$$
$$-2\pi\mathrm{i}\int_{D^{2n+1}} \Lambda^{\mathrm{L-R}}_{2n+1}(u^{-1}\,\mathrm{d}u, 0) \tag{5.30}$$

in which ρ_{2n} can be read off from (4.50), while $\tilde{\rho}_{2n}$ is given by

$$\tilde{\rho}_{2n} = (n+1)n\int_0^1 \mathrm{d}S_+ \int_0^{1-S_+} \mathrm{d}S_-\ \mathrm{str}\{(u\,\mathrm{d}u^{-1})A^{(+)}F_{\tilde{Z}}^{n-1}\} \tag{5.31}$$

with

$$\tilde{Z} = S_+A^{(+)} + S_-u\,\mathrm{d}u^{-1} \qquad F_{\tilde{Z}} = \mathrm{d}\tilde{Z} + \tilde{Z}^2 \tag{5.32}$$

and $\Lambda^{\mathrm{L-R}}_{2n+1}(u^{-1}\,\mathrm{d}u, 0)$ can be read off from (4.19), (4.11) and (4.2), i.e.

$$\Lambda^{\mathrm{L-R}}_{2n+1}(u^{-1}\,\mathrm{d}u, 0) = (-1)^n K_{n+1} B(n+1, n+1)(n+1)\,\mathrm{tr}(u^{-1}\,\mathrm{d}u)^{2n+1}.$$

This result coincides with that of Mañes (1985) where the gauged Wess-Zumino action is constructed in a differential geometric way.

6. Summary

The discussions made in I and this paper show the following characters of this reformulated path integral scheme for deriving anomalies.

(i) We adopt a comoving representation of the Dirac operator, in which the anomalous factor can be understood as the product of the additional phase factors of the diagonalised matrix elements acquired under chiral rotation and gauge transformation, and the anomalous effective action can be regarded as an accumulating effect of the additional argument change of these elements.

(ii) The Wess-Zumino consistent conditions which imposed on the anomalous factor implies that only the arguments of the zero modes have non-trivial topological contributions to the anomaly. This leads to the null space regularisation scheme in which only the argument change of the zero modes is preserved.

(iii) Both Hermitian and non-Hermitian Dirac operators can be treated on equal footing. Likewise, both chiral anomaly and gauge anomaly can be derived in the same way.

(iv) We can also derive the exterior differential formula of anomalies within this scheme. Therefore, the differential geometric objects and techniques can be naturally applied to the investigation of gauged Wess-Zumino effective action in the path integral formalism.

The readers may notice that the essence of our treatment lies crucially on the phase factor of the diagonalised matrix element for a Dirac operator. It is this phase factor which plays the central role in this approach. We believe that it is closely related to the Berry phase (Berry 1984, Sonoda 1986) in the Hamiltonian interpretation of anomalies (Faddeev 1984, Faddeev and Shatashvili 1986). Nevertheless, further investigation is needed.

Acknowledgments

We would like to thank Professors Su-Qing Chen, Han-ying Guo, Bo-yu Hou, Bo-yuan Hou and Xing-chang Song for helpful discussions. This work is partly supported by NSF of China under grant no KR 12040.

References

Alvarez-Gaumé L and Ginsparg P 1984 *Nucl. Phys.* B **243** 449
—— 1985 *Ann. Phys., NY* **161** 423
Atiyah M F and Singer U M 1968 *Ann. Math.* **87** 487, 546
Bardeen W A 1969 *Phys. Rev.* **184** 1848
Berry M V 1984 *Proc. R. Soc.* A **392** 45
Chou K C, Guo H Y K and Song X C 1984 *Phys. Lett.* **134B** 57
Faddeev L D 1984 *Phys. Lett.* **145B** 81
Faddeev L D and Shatashvili S L 1986 *Phys. Lett.* **167B** 225
Fujikawa K 1979 *Phys. Rev. Lett.* **42** 1195
—— 1980a *Phys. Rev. Lett.* **44** 1733
—— 1980b *Phys. Rev.* D **21** 2848
Gipson J M 1986 *Phys. Rev.* D **33** 1061
Ingermanson 1985 *Nucl. Phys.* B **249** 611
Jackiw R 1985 *Symp. on Anomalies, Geometry, Topology* ed W A Bardeen and A R White (Singapore: World Scientific) p 153
Kawai H and Tye S H H 1983 *Phys. Lett.* **140B** 403
Mañes J L 1985 *Nucl. Phys.* B **250** 661
Song X C 1986 *Ann. Phys., NY* **171** 364
Sonoda H 1986 *Nucl. Phys.* B **266** 410
Wang R T and Ni G J 1987 *J. Phys. A: Math. Gen.* **20** 5849
Wess J and Zumino B 1971 *Phys. Lett.* **37B** 95
Witten E 1983 *Nucl. Phys.* B **233** 422
Zumino B 1985 *Symp. on Anomalies, Geometry, Topology* ed W A Bardeen and A R White (Singapore: World Scientific) p 111
Zumino B, Wu Y S and Zee A 1984 *Nucl. Phys.* B **239** 477